

Electromagnetic Duality and $SU(3)$ Monopoles

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ABSTRACT

We consider the low-energy dynamics of a pair of distinct fundamental monopoles that arise in the $N = 4$ supersymmetric $SU(3)$ Yang-Mills theory broken to $U(1) \times U(1)$. Both the long distance interactions and the short distance behavior indicate that the moduli space is $R^3 \times (R^1 \times \mathcal{M}_0)/Z$ where \mathcal{M}_0 is the smooth Taub-NUT manifold, and we confirm this rigorously. By examining harmonic forms on the moduli space, we find a threshold bound state of two monopoles with a tower of BPS dyonic states built on it, as required by Montonen-Olive duality. We also present a conjecture for the metric of the moduli space for any number of distinct fundamental monopoles for an arbitrary gauge group.

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The longstanding conjecture [1] that certain supersymmetric field theories possess an electric-magnetic duality has received further support in recent years from the calculation of dyon spectra. In theories with spontaneously broken $SU(2)$ symmetry, calculations based on a knowledge of the two-monopole moduli space have verified that the spectrum of particle states carrying two units of magnetic charge is consistent with the duality conjecture both in an $N = 4$ [2] and an $N = 2$ theory [3]. The extension of these results to higher magnetic charge has also been discussed [4].

In this note, we consider the extension of this analysis to larger groups. We concentrate in particular on the case of an $N = 4$ supersymmetric Yang-Mills theory with an arbitrary gauge G of rank n that is broken by the adjoint representation Higgs field to its Cartan subgroup, $(U(1))^n$. Even in the purely magnetic sector, such theories present a challenge for the duality conjecture. Their particle spectrum contains $\dim G - n$ electrically charged massive vector mesons, one for each root of the Lie algebra. The most obvious candidates for the dual states are those corresponding to the spherically symmetric classical solutions obtained by imbedding the $SU(2)$ monopole and antimonopole using the subgroups defined by the various roots. However, counting of the zero modes about these solutions suggests that only n of these, corresponding to a special set of simple roots, should be understood as fundamental monopoles and that the remainder should be interpreted as superpositions of these fundamental solitons [5].

As a specific example, consider an $SU(3)$ theory broken to $U(1) \times U(1)$. If we denote the Higgs expectation value in a unitary gauge by a vector Φ_i in the Cartan subalgebra with generators H_i , there is a unique pair of simple roots α and β such that $\alpha \cdot \Phi$ and $\beta \cdot \Phi$ are both positive. There is also a third positive root, $\gamma = \alpha + \beta$. Associated with these roots are electrically charged vector mesons of masses $\alpha \cdot \Phi$, $\beta \cdot \Phi$, and $\gamma \cdot \Phi$. On the magnetic side, there are spherically symmetric classical solutions with magnetic charges α^* , β^* , and γ^* (where $\alpha^* \equiv \alpha/\alpha^2$, etc.). The first two of these each have four zero modes, associated with spatial translations and a $U(1)$ rotation. In contrast, the third, and indeed any BPS [6] solution with magnetic charge γ^* , has eight zero modes and seems to be just one member of a large family of two-particle solutions containing one α^* and one β^* monopole.

This suggests that the dual counterpart of the γ vector boson should be a threshold bound state of an α^* and a β^* monopole. A similar phenomenon was recently found in an $N = 2$ $SU(2)$

theory [3], where the BPS monopole is actually dual to a quark in the fundamental representation. The dual of the massive charged vector boson is a bound state found by quantizing the low energy dynamics of two monopoles as a supersymmetric sigma model whose target space is the two-monopole moduli space.

The obvious difficulty in pursuing this approach in the $SU(3)$ case has been that the moduli space was previously known only for two monopoles of identical charge [7]. However, it turns out not to be too difficult to determine the moduli space for a pair of monopoles with charges α^* and β^* . We will present here a schematic derivation; more details will be given in Ref. [8].

Since we can factor out the center of mass coordinates and a $U(1)$ phase, the moduli space is isometrically decomposable in the form [8],

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{\mathcal{G}} \quad (1)$$

The R^1 is parametrized by a coordinate χ whose conjugate momentum P_χ we will later identify with a “total” electric charge. Unless the mass-ratio of the two monopoles is rational, χ is not in general a periodic coordinate. The identification by \mathcal{G} is closely tied to the quantization of dyons, and we will return to the matter of this infinite discrete group \mathcal{G} later.

A natural starting point for determining \mathcal{M}_0 is to study the interactions of two widely separated dyons. For the $SU(2)$ theory, Manton showed that these forces imply that the metric of the relative coordinate moduli space asymptotically approached a Taub-NUT metric with negative length parameter [9]. This metric has a short distance singularity that renders it unacceptable as the moduli space. But even apart from this singularity, symmetry arguments alone show that the two metric cannot be identical. The Taub-NUT manifold has an exact $U(1)$ isometry that would correspond to a rotational symmetry about the axis through the center of the two monopoles. In the actual solutions, the interaction between the monopole cores gives rise to terms that break this symmetry, although their magnitude falls exponentially with the monopole separation [7].

The long-range forces between the two $SU(3)$ monopoles are similar to those in the $SU(2)$ case, except that the product of charges (magnetic, electric, or scalar) which enters each of these forces gives a factor of $\alpha \cdot \beta = -\alpha^2/2$. Because of this negative sign, \mathcal{M}_0 asymptotically approaches a nonsingular Taub-NUT metric with positive length parameter. In this case, however, symmetry considerations do not exclude the possibility of an exact identity, since the $SU(3)$ solutions with

separated monopoles have an exact axial symmetry, a property that can be traced to the presence of the second unbroken $U(1)$. Furthermore, the Taub-NUT manifold with positive length parameter has a fixed point $r = 0$ under its rotational $SU(2)$ symmetry. That could plausibly correspond to the spherically symmetric γ^* monopole. Note in particular that the zero mode structure about this solution agrees precisely with what one would expect from the Taub-NUT metric in the neighborhood of its fixed point [8].

To verify that \mathcal{M}_0 is in fact a Taub-NUT manifold, we use the fact that the moduli space metric must be hyperkähler [7], so that \mathcal{M}_0 must be a four-dimensional manifold with a rotationally invariant anti-self-dual metric [10]. Any such metric can be written in the form [11]

$$g_0 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2, \quad (2)$$

where the line elements σ_i 's obey $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$, while

$$\frac{2bc}{f} \frac{da}{dr} = b^2 + c^2 - a^2 - 2\lambda bc, \quad \text{and cyclic permutations thereof,} \quad (3)$$

where either $\lambda = 1$ or $\lambda = 0$. The case of $\lambda = 1$ was studied in detail by Atiyah and Hitchin, who showed that the only three possibilities (up to irrelevant permutations of σ_k 's) are:

- 1) $a = b = c$ case: the flat R^4 .
- 2) $a = b \neq c$ case: the Taub-NUT geometry with positive length parameter.
- 3) $a \neq b \neq c$ case: the Atiyah-Hitchin geometry

The case of $\lambda = 0$ is even simpler and leads to only one new possibility.

- 4) the Eguchi-Hanson gravitational instanton [12].

Only the second of these approaches the correct asymptotic geometry. Therefore, \mathcal{M}_0 must be given by the smooth Taub-NUT geometry:

$$g^{(4)} = \left(1 + \frac{2l}{r}\right) (dr^2 + r^2 \sigma_1^2 + r^2 \sigma_2^2) + \frac{4l^2 \sigma_3^2}{1 + 2l/r}, \quad (4)$$

where the length parameter l is positive and inversely proportional to the reduced mass of the two monopoles.

With an appropriate choice of the angular coordinates, the sum $\sigma_1^2 + \sigma_2^2$ is the line element of a two-sphere $d\theta^2 + \sin^2 \theta d\phi^2$ and is independent of the third angular coordinate which we call ψ . The vector field $\xi_3 = \partial/\partial\psi$ dual to σ_3 thus become a Killing vector field and in fact generates the extra axial symmetry alluded to earlier. The ranges of θ and ϕ are $[0, \pi]$ and $[0, 2\pi]$ respectively, while the range of ψ depends on whether the rotational isometry is $SU(2)$ or $SO(3)$. A careful comparison with the long range interactions of the two monopole reveals that the conserved momentum P_ψ is simply the $U(1)$ generator $(\alpha \cdot H - \beta \cdot H)/3$ which admits half-integer eigenvalues, implying that $\psi \in [0, 4\pi]$. The full isometry group of \mathcal{M}_0 is thus $SU(2) \times U(1)$.

How do we determine the discrete subgroup \mathcal{G} ? It is most instructive to consider the Hamiltonian \mathcal{H} of the two monopoles at large separations. In particular we want to concentrate on the internal motions that generate electric charges Q_α and Q_β of the respective monopoles [8]:

$$\mathcal{H} = \frac{1}{2}m_\alpha Q_\alpha^2 + \frac{1}{2}m_\beta Q_\beta^2 + \mathcal{H}_{\text{rel}}\left(\frac{Q_\alpha - Q_\beta}{2}; r\right), \quad (5)$$

where \mathcal{H}_{rel} encodes interaction terms that vanish at large separation, while m_α and m_β are the masses of the respective monopoles of magnetic charge α^* and β^* . The electric charges are integer-quantized and thus can be realized as the canonical momenta of cyclic variables ξ_α and ξ_β of period 2π , which are coordinates in the moduli space \mathcal{M} .

Note that the interaction Hamiltonian \mathcal{H}_{rel} depends only on the “relative” electric charge $Q_\alpha - Q_\beta$. Rearranging the Hamiltonian to factor out the noninteracting part, we find

$$\mathcal{H} = \frac{1}{2}(m_\alpha + m_\beta) \left(\frac{m_\alpha Q_\alpha + m_\beta Q_\beta}{m_\alpha + m_\beta} \right)^2 + 2 \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \left(\frac{Q_\alpha - Q_\beta}{2} \right)^2 + \mathcal{H}_{\text{rel}}\left(\frac{Q_\alpha - Q_\beta}{2}; r\right). \quad (6)$$

Now it is easy to see that we must identify the “total” and “relative” charges with the conjugate momenta of the coordinate χ and ψ introduced above,

$$P_\chi = \frac{m_\alpha Q_\alpha + m_\beta Q_\beta}{m_\alpha + m_\beta}, \quad P_\psi = \frac{Q_\alpha - Q_\beta}{2}. \quad (7)$$

Accordingly, χ and ψ are related to the ξ ’s by

$$\chi = \xi_\alpha + \xi_\beta, \quad \psi = 2 \frac{m_\beta \xi_\alpha - m_\alpha \xi_\beta}{m_\alpha + m_\beta}. \quad (8)$$

Since the ξ ’s are periodic in 2π , the following identifications on χ and ψ are necessary,

$$(\chi, \psi) = (\chi, \psi + 4\pi), \quad (\chi, \psi) = (\chi + 2\pi, \psi + \frac{4m_\beta}{m_\alpha + m_\beta}\pi). \quad (9)$$

The first simply reasserts the fact that ψ has period 4π , while the second generates the discrete group $\mathcal{G} = Z$. For equal masses $m_\alpha = m_\beta$, in particular, χ is also periodic in 4π so that the moduli space can be written in the form,

$$\mathcal{M} = R^3 \times \frac{S^1 \times \mathcal{M}_0}{Z_2}. \quad (10)$$

But unless the mass-ratio is rational, the R^1 factor need not collapse to a circle by the action of the group $\mathcal{G} = Z$, and we have in general,

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z}. \quad (11)$$

Once the moduli space is known, the quantization of the low energy dynamics is straightforward. In fact, for the purpose we have in mind, we will not even need to write down the explicit form of the Lagrangian. From the hyperkähler property of the moduli space and the counting of fermionic zero modes, we already know that the quantum mechanics on \mathcal{M} must be $N = 4$ supersymmetric with sixteen fermionic coordinates [13]. According to Witten [14], then, all zero energy states of this theory are in one-to-one correspondence with harmonic forms on the moduli space.

On the other hand, there must be one and only one zero energy bound state of the two fundamental monopoles that carries no electric charge. This means that the duality hypothesis checks out only if there is a unique normalizable harmonic form on $(R^1 \times \mathcal{M}_0)/Z$ invariant under the isometries induced by $\partial/\partial\chi$ and $\partial/\partial\psi$. The first condition can be achieved trivially by requiring that the harmonic form does not depend on χ at all. Furthermore since a normalizable harmonic p -form on the k -dimensional manifold produces a normalizable harmonic $(k - p)$ -form via the Hodge dual procedure, the unique bound state must come from either a self-dual or an anti-self-dual 2-form on \mathcal{M}_0 . (This fact was previously emphasized by Sen [2].)

In terms of four orthonormal 1-forms,

$$w^0 = -\sqrt{1 + \frac{2l}{r}} dr, \quad w^1 = r\sqrt{1 + \frac{2l}{r}} \sigma_1, \quad w^2 = r\sqrt{1 + \frac{2l}{r}} \sigma_2, \quad w^3 = \frac{2l}{\sqrt{1 + 2l/r}} \sigma_3, \quad (12)$$

the most general self-dual and anti-self-dual 2-forms can be written as

$$F_i^{(\pm)} \left(2w^0 \wedge w^i \pm \epsilon_{ijk} w^j \wedge w^k \right) \quad (13)$$

Requiring that the exterior derivative vanishes, which is sufficient for showing that they are harmonic forms, tells us that $F_i^{(\pm)}$ are functions only of r obeying a first order differential equation

[2]. It is then a straightforward exercise to show that only $F_3^{(-)}$ produces a normalizable and nonsingular harmonic 2-form,

$$\begin{aligned}\Omega_3^{(-)} &= -\frac{1}{(r+2l)^2} (w^0 \wedge w^3 - w^1 \wedge w^2) \\ &= \frac{2l}{(r+2l)^2} dr \wedge \sigma_3 + \frac{r}{r+2l} \sigma_1 \wedge \sigma_2.\end{aligned}\tag{14}$$

As noted above, this implies the existence of a bound state of the two fundamental monopoles, which must have the magnetic charge γ^* . This bound state is intrinsic to the supersymmetric theory and exists only because the underlying Yang-Mills theory has four global supersymmetries. The 2-form $\Omega_3^{(-)}$ is invariant under the $U(1)$ translation generated by the Killing vector field $\partial/\partial\psi$, as we anticipated, not to mention being invariant under the spatial rotation group $SU(2)$. One may regard this bound state as the quantum analogue of the spherically symmetric γ^* monopole found in Ref. [5].

In fact, this 2-form generates more than the purely magnetic bound state of two monopoles. Rather than requiring the state to be independent of χ , we could try to construct the states with $P_\chi \neq 0$ by tensoring the wavefunction with $\exp(iP_\chi\chi)$. Since these states are invariant under $\partial/\partial\psi$, Q_α is equal to Q_β , and thus the “total” charge P_χ is the integer $Q_\alpha = Q_\beta$. Furthermore, in this sector of trivial P_ψ , the “total” charge P_χ can be identified with the eigenvalues of $\gamma \cdot H$. Therefore, the harmonic 2-form $\Omega_3^{(-)}$ indeed generates the whole tower of dyonic states with magnetic charge γ^* and integral electric charge $n\gamma$, as required by Montonen-Olive duality conjecture. This provides a nice consistency check on our moduli space \mathcal{M} , and specifically the division by $\mathcal{G} = \mathbb{Z}$.

Finally, we must show that $\Omega_3^{(-)}$ is indeed the unique normalizable harmonic form on \mathcal{M}_0 . Otherwise there would be a superfluous bound state inconsistent with the duality conjecture. For the purpose let us consider the following action of the Clifford algebra generated by Γ^a ’s on the space of forms $\Lambda^*(\mathcal{M}_0)$,

$$\Gamma^a(w^b) = w^a \wedge w^b - \delta^{ab}, \quad \Gamma^a(w^b \wedge V) = \Gamma^a(w^b) \wedge V - w^b \wedge \Gamma^a(V), \quad V \text{ is an arbitrary } p\text{-form.}\tag{15}$$

It is easy to see that this does represent the Clifford algebra with $\{\Gamma^a, \Gamma^b\} = -2\delta^{ab}$; furthermore, it is well known that the resulting Dirac operator on $\Lambda^*(\mathcal{M}_0)$ is a sum of the natural exterior

derivative d and its adjoint [15]:

$$\Gamma^a \nabla_a = d + d^\dagger \quad (16)$$

Thus a p -form is harmonic if and only if it is also a zero mode of such a Dirac operator. Squaring the Dirac operator, we find

$$\Gamma^a \Gamma^b \nabla_a \nabla_b = -\nabla^a \nabla_a + \frac{1}{4} [\Gamma^a, \Gamma^b] [\nabla_a, \nabla_b] = -\nabla^a \nabla_a + \mathcal{R}. \quad (17)$$

On 0-forms, the curvature piece \mathcal{R} is trivial as usual, while on 1-forms $V_a w^a$, it reduces to a matrix multiplication by the Ricci tensor, $\mathcal{R}V_a = R_a^b V_b$. This also vanishes because the Taub-NUT space is Ricci-flat. On 2-forms $V_{ab} w^a \wedge w^b$, after taking into account the vanishing Ricci tensor, the action of \mathcal{R} is easily found to be

$$\mathcal{R}V_{ab} = -R_{ab}{}^{cd} V_{cd}. \quad (18)$$

The anti-self-dual nature of the curvature tells us that this potential is nontrivial only when V is anti-self-dual as well. Finally, we do not need to consider 3-forms and 4-forms separately since the Hodge dual procedure maps all normalizable harmonic p -forms to $(4-p)$ -forms bijectively. Therefore, with the possible exception of anti-self-dual 2-forms, all harmonic forms on \mathcal{M}_0 must be zero modes of the covariant Laplacian $-\nabla^a \nabla_a$.

Suppose a p -form V is a zero mode of this Laplacian. By taking the inner product of V and $-\nabla^a \nabla_a V$ and equating the result to zero, we find

$$0 = - \int \langle V, \nabla^2 V \rangle = \int \langle \nabla V, \nabla V \rangle - \oint \langle V, \nabla V \rangle \quad (19)$$

where $\langle \cdot, \cdot \rangle$ is the natural pointwise inner product of tensors, defined by the metric. Since $\nabla V \equiv 0$ implies a constant $\langle V, V \rangle$, leading to either an infinite norm or an identically vanishing V , a normalizable and nontrivial solution requires that the boundary term be nonzero. The boundary volume element grows as r^2 at the asymptotic infinity, and at least one orthonormal component of V must have tail vanishing no faster than $1/r$. Such an asymptotic behavior again leads to a divergent norm of V itself so the solution is still unacceptable. Therefore $\Omega_3^{(-)}$ is the unique normalizable harmonic form on \mathcal{M}_0 .

The previous analysis also explains why the normalizable harmonic form $\Omega_3^{(-)}$ exists. The potential is diagonalized with respect to the decomposition of an arbitrary anti-self-dual 2-form V

into three pieces $V_i^{(-)} = (2V_{0i} - \epsilon_{ijk}V_{jk})$, and the diagonal entries of the potential, $U_i^{(-)} = -R_{0i0i}$ (no summation on i), are found to be

$$U_1^{(-)} = U_2^{(-)} = +\frac{l}{(r+2l)^3}, \quad U_3^{(-)} = -\frac{2l}{(r+2l)^3}. \quad (20)$$

The potential is attractive only along $V_3^{(-)}$ direction precisely where we discovered the harmonic form $\Omega_3^{(-)}$ earlier.

These results can be immediately extended to any group G of rank n broken to $U(1)^n$. If the n preferred simple positive roots are $\alpha_i, i = 1, \dots, n$, the moduli space of a pair of fundamental monopoles with charges α_i^* and α_j^* is again determined by the smooth Taub-NUT geometry whenever $\alpha_i^* \cdot \alpha_j^* < 0$, although the action of the discrete group \mathcal{G} could be modified if Lie algebra of G is not simply-laced [8]. The bound states of the total magnetic charge $\alpha_i^* + \alpha_j^*$ should follow similarly from our construction.

Furthermore, we believe that the moduli space of m ($\leq n$) distinct fundamental monopoles should be given by a simple and tractable generalization of the $m = 2$ case studied here. We have seen that the moduli space of two fundamental monopoles is obtained from its long distance behavior alone; it is very likely that the same is true of the moduli space of these m monopoles. The asymptotic form of this space can be found by a procedure analogous to that of Ref. [16].

This leads us to conjecture that the metric of the moduli space of our m distinct fundamental monopoles is

$$g^{(4m)} = M_{jk} d\mathbf{r}_j \cdot d\mathbf{r}_k + \kappa M_{jk}^{(-1)} (d\xi_j + W_j)(d\xi_k + W_k) \quad (21)$$

where \mathbf{r}_j is the Cartesian coordinate of the j -th monopole and W_j is a weighted sum of $U(1)$ vector potential 1-forms due to other monopoles, i.e., $W_j = \sum_{k \neq j} \alpha_k^* \cdot \alpha_j^* \mathbf{A}_k^{(j)} \cdot (d\mathbf{r}_k - d\mathbf{r}_j)$ with $\mathbf{A}_k^{(j)}$ being the Dirac potential of the k -th monopole evaluated at \mathbf{r}_j . Also $\kappa = g^4/(4\pi)^2$ where we restored the magnetic coupling g explicitly. The coordinate ξ_j plays a role similar to those of ξ_α and ξ_β above, and has periodicity $2\pi/\alpha_j^2$. Note that we have not factored out the center-of-mass coordinate. Some of the long range interaction is encoded in the $m \times m$ matrix M_{jk} which must have the form

$$M_{jj} = m_j - \sum_{k \neq j} \frac{g^2 \alpha_k^* \cdot \alpha_j^*}{4\pi |\mathbf{r}_j - \mathbf{r}_k|}, \quad M_{jk} = M_{kj} = \frac{g^2 \alpha_k^* \cdot \alpha_j^*}{4\pi |\mathbf{r}_j - \mathbf{r}_k|}, \quad j \neq k, \quad (22)$$

where m_j 's are the BPS masses of the monopoles. It is important to note that we require all monopoles to be fundamental and distinct from one another so that $\alpha_i^* \cdot \alpha_j^* \leq 0$ whenever $i \neq j$. The simplest case, $m = 2$ with $(\alpha_1 \cdot \alpha_2)^2 / \alpha_1^2 \alpha_2^2 = 1/4$, is easily shown to reproduce our moduli space \mathcal{M} .

While we have not shown that this is the right metric for the moduli space at all separations, there are some indications that it is: Apart from being hyperkähler, this geometry has the right symmetry properties. The overall rotation of the m monopole configuration is still an isometry since the metric coefficients M_{jk} are dependent only on the relative distances. The m axial rotations of $\xi_j \rightarrow \xi_j + \text{const}$ are exactly those expected from the surviving $U(1)^n$ gauge group. Finally the geometry is smooth wherever two of \mathbf{r}_j coincide.

If this conjecture is correct, it should be a matter of straightforward, albeit tedious, algebra to find the appropriate (anti)-self-dual $2(m-1)$ -form on the analog of \mathcal{M}_0 that represents the unique magnetic bound state of charge $\sum \alpha_j^*$ and the whole dyonic tower thereof. This problem is left for a future study.

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As this work was completed, we learned of two related papers, one by Gauntlett and Lowe and another by Connell [17].

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